SOLUTIONS OF PROBLEMS.—ALGEBRA.

Binary cubic, f, $\Delta=(f_1, f)_2$, the second transvectant of f over f; and $R=(\Delta_1\Delta)_2$. Then $\Delta_{\kappa\lambda}=(\kappa^2-\frac{1}{2}\lambda^2 R)\Delta$ where $\Delta_{\kappa\lambda}$ is the Δ covariant for the cubic pencil $\kappa f+\lambda Q$, Q being first transvectant of f over Δ. 255 ---------- 58

Binary form, to be perfect nth power its Hessian must vanish. 260 ---------- 105

Binary quartic, express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ in terms of roots of the quartic. 264 ---------- 147

Biquadratic, obtain reduced cubic of. 265 ---------- 161

Boats, two ferry, start from opposite sides of river, etc. 269 ---------- 208

e and π, express as infinite continued fractions. 263 ---------- 145-147

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[(a^2y^2+b^2x^2)^{\frac{1}{2}}+ab]^2$. 250 ---------- 36

Find simplest integral form of sum $y(y-1)\ldots y(y-x)\ldots zy(zy-x)$. No solution received. 270 ---------- 208

Find simplest integral form of sum of $y(y-1)\ldots(y-x)+2y(2y-1)\ldots(2y-x)+\ldots+zy(zy-1)\ldots(zy-x)$. 271 ---------- 228

Find sum to n terms of $1+2/n+n(n-2)/2.4+\ldots$ 247 ---------- 13

Functions, hyperbolic, express as infinite continued fractions. 268 ---------- 185-187

Pasture, rented by A, B, C, for fixed amount, etc. 256 ---------- 80

Prove that $\pi^{16}[1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\ldots]$ times certain factors. 248 ---------- 14

Prove that $\frac{1}{n+1}+\frac{1}{2(n+2)}+\ldots=\frac{1}{n^2}+\frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\cdots\frac{1}{l(n-l)}\right]$ l being equal to $n-1$, $n>1$. 251 ---------- 37

Prove that certain relations between the real finite quantities x, a, b, r, s, λ, μ requires that $x^2=a^2+b^2$. 272 ---------- 228-229

Triangle, given area 420, sides 21, 61, 65. Assuming area correct, determine error in sides. 259 ---------- 104

Trigonometric functions, express as infinite continued fractions. 267 ---------- 184-187

Series, sum $a/b+a^3/3b^3+a^5/5b^5+\ldots\infty$, $b>a$. 271 ---------- 209

Series, $1+3+7+17+\ldots$, find nth term and sum of n terms. 266 ---------- 161-163

Series, sum to infinity $n^2/(4n^2-1)^2$, beginning with $n=1$, n being always odd. 258 ---------- 83

Series, sum to infinity $n/(4n^2-1)^2$, beginning with $n=1$, n being always odd. 262 ---------- 145

Series, sum to infinity $1/n^p+3/n^{2p}+5/n^{3p}+\ldots$ 261 ---------- 105

Series, sum to infinity $n^2/(16n^2-1)^2$ beginning with $n=1$. 254 ---------- 58

Solve, (1) $x-y=\frac{1}{3}\pi$, (2) $\sin x=\cos^3 y$. 252 ---------- 38

Solve $x+y+z=5$, $x^2+y^2=z^2$, $x^3+y^3+z^3=8$. 249 ---------- 15

Solve (1) $x+y=10$, (2) $3x=\log_{10}y$. 257 ---------- 81

Solve $a_1^n+a_2^n+a_3^n+\ldots+a_r^n=A^n$, $a_1^m+a_2^m+a_3^m+\ldots+a_r^m>$ or $<A^m$, according as $m<$ or $>n$, provided, etc. 229 ---------- 227-228

AVERAGE AND PROBABILITY.

Chords drawn in every possible direction through circumference of circle, find average length. 179 ---- 234

Circular arcs, average length of, having center at corner of square, etc. 172 ---- 192-193 233

Circle, chords drawn through every point of. Find average length of. 175 (incorrectly numbered 174) ---- 39

Cube cut at random by a plane, what is chance that the section is hexagon? 177 (incorrectly numbered 176) ---- 233

Ellipse, chord c, drawn at random in given; find average area of segment cut off. 174 (incorrectly numbered 173) ---- 217-218

Line l divided into three equal parts, find mean value of triangle formed of them. 176 (incorrectly numbered 175) ---- 39

Line l divided into n parts, find mean value of product of p of the n segments. 167 ---- 192 216-217

Planes, two random, cut a given sphere, what is chance that they intersect within the sphere? No solution received. 178 ---- 234

Triangle, find average length of all straight lines drawn within. 173 (incorrectly numbered 172) ---- 15

Triangle, find average area of, formed by taking three points at random in sides of regular n-gon. No solution received. 163 ---- 216

Triangle, find mean value of part of, containing center of gravity, triangle being divided by line passing through two points taken at random within it. No solution received. 160 ---- 215

Triangle, find area of, formed by drawing line at random through each of three random points in given triangle. No solution received. 170 217

CALCULUS.

Cone, thread makes n equidistant turns around rough; find distance bird flies in unwinding thread. 209 ---- 107-110

Differential equation, $d^2y/dx^2=axy$ to solve. 229 ---- 230

Differential equation, certain, to transform so that variables are separable. 215 ---- 59

Integral, definite, note on problem 207 ---- 17

Integral $\int \sqrt{(1+y)/(1+y^2)}\,dy$, to evaluate. 133 ---- 83-84

Integral, $\int_0^{\frac{1}{2}\pi} x\,dx/\sin x[1+.16\cos^2 x]^{\frac{1}{2}}$, to evaluate. 217 ---- 84-85

Integrals, (*a*) $\int_0^{\frac{1}{2}\pi}[\sin mx\,\sin nx/\sin x]dx$, (*b*) $\int_0^{\frac{1}{2}\pi}\cos mx\,\sin nx\,dx/\sin x$. 220 (incorrectly numbered 219) ---- 163-165

Integral, $\int_0^1 (1+x^m)^n dy dx$, to evaluate. 223 ---- 190

Integral, $\int_0^\infty \tan^{-1}(\tan\alpha\,\sin x)dx/x=\frac{1}{2}\pi\log(\tan\alpha+\sec\alpha)$, prove. 227 ---- 213-214

If $x=v^{1/v-1}$, what is $f(x)$ such that $v=f(x)$. 211 ---- 18
If $f(x)$ is any function of x, $f'(x)$ its derivative, $u=[f'(x)]^{-\frac{1}{2}}$, etc. 213 39-40
Limit, find, of $(1/n)[(n+1)(n+2)...(2n)]^{1/n}$ as $n \doteq \infty$. 218 (incorrectly numbered 219) ---- 85-87
Limit, find, of, $\tan^{-1}x\log x$ as $x \doteq \infty$. 224 ---- 190
Limit, show that, of $\sum_{\lambda=1}^{\lambda=n} (\lambda^k/n^{k+1}) = \frac{1}{k+1}$ as $n \doteq \infty$. 223 ---- 211-212
Polynomial $f(x)$ determine, entirely in x and of 7th degree, etc. 210 -- 17-18
π^2, prove, $=6[2^2/(2^2-1)-3^2/(3^2-1)-5^2/(5^2-1)...]$. 214 ---- 214-215
Pyramids, of all triangular, standing on given base, to find that whose surface is least. 220 (incorrectly numbered 219) ---- 85-87
Polygon of n sides, to determine least described about a given circle. 221 (incorrectly numbered 220) ---- 148
Prove $16/\pi^2=(1+1/a^2)(1+1/b^2)(1+1/c^2)...$ where a, b, c, ... are consecutive primes. 222 ---- 190 211
Prove that $s_3+s_4=\log 3$, etc., if $s_n=2(1/n-2/2n^3+1/5n^5+1/7n^7-2/9n^9+...)$. 223 ---- 190-191
Series, to find limit of sum of certain. 216 ---- 84
Sphere, radius r, dropped into conical vessel, to find contents by given formula. 228 ---- 230

DIOPHANTINE ANALYSIS.

Congruence, how many solutions has the, $\lambda+\mu+\nu \equiv 0 \pmod{p-1}$, etc. 132 ---- 59-60
Eight persons wish to play a series of games of progressive duplicate whist. In one evening, twelve boards are played, etc. 126 ---- 110-114
Equation $2x^2+2x+1=\square=u^2$, show that u is always the sum of two squares. 136 ---- 232
From the numbers 0, 1, 2, 3, ..., 42, select seven, such that the forty-two differences, etc. 132 ---- 215
Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2$, 3, 4. 138 (incorrectly numbered 137) ---- 165
$7x^2-111=y^2$. Required a value of, for $y>1$ which shall be a prime integer. 137 (incorrectly numbered 136) ---- 148-149 191

GEOMETRY.

Angle, trisect by means of a tractrix. 300. No solution received ---- 229
Bisector of angles of triangle, the greatest bisector falls on least side; prove without aid of circle. 286 (incorrectly numbered 285) --- 115-116
Circle, at end of uniformly revolving radius of, a line equal to diameter is pivoted, etc. 290 (incorrectly numbered 289). No solution received ---- 185
Circle, from point on given, to draw two chords so that chord PA: chord

Circle on any focal radius of ellipse touches auxiliary circle. 299 (incorrectly numbered 297) ---------- 210-211
$PB=m:n$. 289 (incorrectly numbered 288) ---------- 187
Circle, variable, touches ellipse, and chord of contact through the other two points of intersection touches a similar co-axial ellipse. Find locus of center of variable circle. 295 (incorrectly numbered 293) 166-167
Ellipse, a point is taken on any diameter of, such that the tangents from it intercept on the tangent, etc. 281 (incorrectly numbered 280) 61-62
Hyperbola drawn touching the axes of an ellipse, and asymptotes of hyperbola touch ellipse. Prove center of hyperbola lies on one of conjugate diameters of ellipse. 276 (incorrectly numbered 275) 20-21
Line, AB, straight, placed between two intersecting lines MN and PQ, etc. 215 (incorrectly numbered 274) ---------- 18-19
Line, indefinite, pivoted to end of revolving radius, etc. 296 (incorrectly numbered 294) ---------- 118
Line perpendicular to each of two intersecting lines, etc., it is assumed that they have a common perpendicular. 282 (incorrectly numbered 281) ---------- 62
Lines $AB=BC$ are perpendicular to each other; E, M their mid-points, etc. 298 (incorrectly numbered 296) ---------- 153
Locus, $x^2y-a^2x+a^2y=0$. Show that points whose abscissae are 0, $a\sqrt{3}$, $-a\sqrt{3}$, are points of inflexion. 288 (incorrectly numbered 287) 116-117
Locus, $x^3+y^3-3xy=0$. Show that (1, 1) is conjugate point. 291 (incorrectly numbered 290) ---------- 149
Locus of $r=a(1+2\cos\theta)$, apply to trisection of angle, etc. 301 ---------- 229-230
Locus, $(x^2+y^2)^3=mx^3$, apply to finding a cube m times a given cube. 294 (incorrectly numbered 292) ---------- 209
Nine points lying by three in three columns and in three rows. Draw through them, etc. 292 (incorrectly numbered 290) ---------- 187-188
Orthogonal system of circles in a plane, prove that every system is an isothermal system. 268 (incorrectly numbered 267) ---------- 18
Pedal line of any point on triangle's circum-circle bisects the distance between this point and ortho-center of triangle. 293 (incorrectly numbered 290) ---------- 166
Pedal lines of any two points on the circum-circle of a triangle, etc. 283 (incorrectly numbered 282) ---------- 62-63
Pedal lines, the right angled intersection of, of any circum-circle, lies on the "nine points circle" of inscribed triangle. 284 (incorrectly numbered 283) ---------- 63
Pentagon, Dürer method of approximate construction, compute error. 277 (incorrectly numbered 276) ---------- 41
Regular polygon inscribed in a circle; tacit assumption that as number of sides is increased in any manner its perimeter has a fixed limit, etc. 278 (incorrectly numbered 277) ---------- 42

Radii R and r whose center is the origin revolve with uniform angular velocities 3θ and θ, etc. 285 (incorrectly numbered 284)------- 114-115
Triangle, find side of maximum similar triangle, etc. 279 (incorrectly numbered 278)---------- 60
Triangle, one side and opposite angle is fixed. Find locus of center of inscribed circle. 297 (incorrectly numbered 295)---------- 188-189
Triangle, on sides of given, measure off equal distances, etc. 287 (incorrectly numbered 286)---------- 116
Triangle, to construct geometrically the maximum equilateral, circumscribed about a given triangle. 280 (incorrectly numbered 279) 42-43
Volume generated by revolving semi-segment of a circle about sine of arc, find by pure geometry. 274 (incorrectly numbered 273)--- 64

GROUP THEORY.

Any group of order $\sum_{i=1}^{n} p_i$ (p_i a prime $\neq p_j$) may be generated as, etc. 14 64-67
Chess tournament, in a, between eight players, etc. 8---------- 150-151
Given $U_1=a'$, $V_1=\beta'$, and the recursion formula $U_y=a'V_{y-1}+a''U_{y-1}$, etc. 12---------- 21
Order of linear homogeneous group of n letters is $(p^{n-1})(p^n-p)\ldots(p^n-p^{n-1})$. Give other proofs than those given by Burnside. 13-- 21-22

MECHANICS.

Axiomatic principles of physics which is equivalent to theorem of compound of two circles. No solution received. 189---------- 214
Particle, acted upon by central force, find path of, etc. 187---------- 231
Particle, find path described by, acted upon by a central force which is proportional to distance of particle. 187---------- 89
Planet, find velocity of, at given point in its orbit. 191---------- 214
Point, P, keeps at uniform distance from, and moves with uniform angular velocity around a point Q which is in harmonic motion, etc. 185 (incorrectly numbered 186)---------- 22
Pole hinged at bottom leans against mid-point of smooth rope, etc. 190 214
Sphere, solid, rolls down trough formed by two planes, etc. 192----- 231-232
Spherical bubbles are rising in water; find relation between radius and velocity. No solution received. 188---------- 214

MISCELLANEOUS.

Ingot of pure gold melted at Mint and 10 ounces taken out and 10 ounces silver added. Process repeated 10 times, etc. 162 (incorrectly numbered 158)---------- 151-152
Numbers, no multiply perfect, of multiplicity n containing only n distinct primes. 160 (incorrectly numbered 156)---------- 44-45

Rod, inelastic, placed with upper end on rough vertical plane, etc. To determine subsequent motion. No solution received. 161 (incorrectly numbered 157)---------- 234
Streams, two straight, of different volumes, etc., come together. Find path of body in mid-point of either. No solution received. 163 234
Vessels, two, containing *a* gallons of alcohol and *b* gallons of water, respectively; *c* gallons are drawn from each, etc. 159 (incorrectly numbered 155)---------- 43-44

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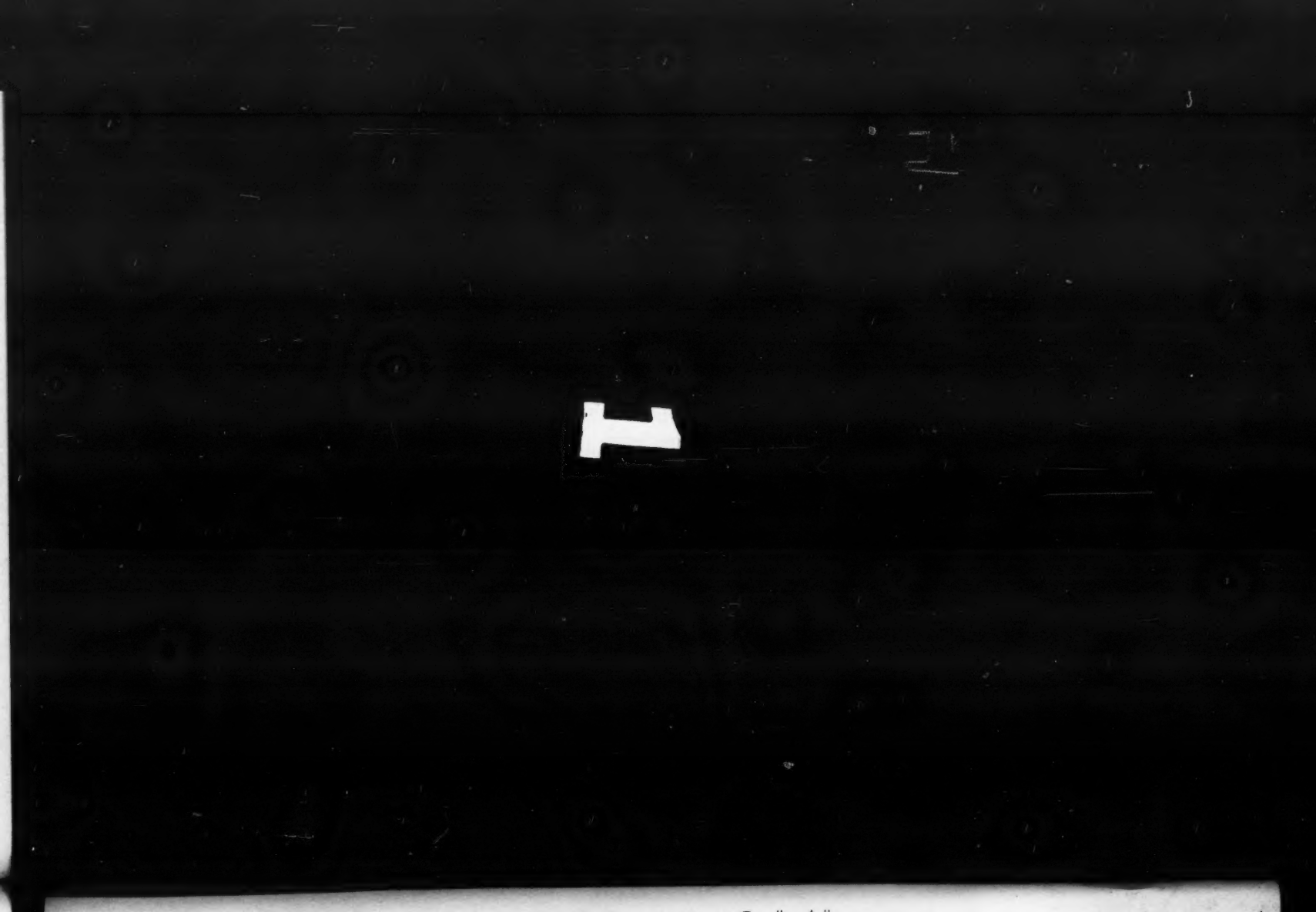

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. JANUARY, 1906. No. 1.

ON THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.

By R. P. BAKER, University of Iowa.

The object of this paper is a systematic account of the subject designated by Muir "vanishing aggregates of determinant products."* It will be seen that they are in general Laplacian expansions of vanishing determinants, which do not vanish when regarded as functions of the minor determinants. They include as a special case the fundamental determinant identity, the vanishing determinant with two identical rows or columns. This case (historically Vandermonde's identity†) underlies all the others. The first example of a more complicated kind is due to Bezout‡, who gave among the determinants of the 6×3 array the relation: $(123)(456)-(124)(356)+(134)(256)-(234)(156)=0$. Desnanot§ gave a number of new forms and the general method of deriving new from the old, called now the method of the extensional. More recently Cayley‖, Sylvester,¶ and Muir° have considered the general forms, while Kronecker,' Runge", and others have studied the special forms relating to an axisymmetric determinant.

* *The Theory of Determinants*, p. 52.
† Ib., p. 22.
‡ Ib., p. 52.
§ Muir, p. 143.
‖ *Coll. Math. Papers*, 1. 55.
¶ *Phil. Mag.*, 1850.
° *Proc. R. S. E.*, 1891, p. 73.
' Sitzungsb, *d. k. Akad. d. Wiss.*, 1882.
" Crelle, 93–319.